Models of Set Theory I – Summer 2017

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Problem 29 [4 points] Let α be an ordinal, let $\mathbb{P} \in M$ be a forcing notion, and let M be a countable ground model. We call $\dot{x}$ a *nice* $\mathbb{P}$*-name* for a subset of α in case $\dot{x}$ is of the form

$$\dot{x} = \bigcup \{\{\check{\beta}\} \times A_\beta \mid \beta \in \alpha\},$$

where each A_β is an antichain of $\mathbb{P}$.

- Show that if $\alpha \in \mathrm{Ord}^M$, and $1 \Vdash \dot{x} \subseteq \check{\alpha}$, then there is a nice name $\dot{y}$ such that $1 \Vdash \dot{x} = \dot{y}$.
- Let $\mathbb{P}$ be Cohen forcing, and let κ be an infinite cardinal in M. Calculate the number of nice $\mathbb{P}$-names for subsets of κ in M. Use this to argue that $(2^\kappa)^{M[G]} = (2^\kappa)^M$.

Problem 30 [4 points] Assume that $r \colon \omega \to 2$ is a Cohen real over the countable ground model M.

- Show that both a and b are Cohen reals over M, where

$$a = \{(n, i) \mid (2n, i) \in r\}, \; b = \{(n, i) \mid (2n+1, i) \in r\}.$$

- Show that if G is generic for Cohen forcing over M, then in $M[G]$, there are $2^{\aleph_0}$-many pairwise different Cohen reals over M.

Problem 31 [3 points] A subset X of $\mathbb{R}$ has *strong measure zero* if for any sequence $\langle n_i \mid i < \omega \rangle$ of natural numbers, it can be covered by a sequence of intervals $\langle I_i \mid i < \omega \rangle$ such that I_i has length at most $\frac{1}{n_i}$ for every $i < \omega$. Let M be a countable ground model. Show that the ground model reals have strong measure zero in any generic extension for Cohen forcing.

Problem 32 [9 points] Let B be a complete Boolean algebra. We shall define a *Boolean-valued model* V^B inductively as follows.

1. $V_0^B = \emptyset$.
2. $V_{\alpha+1}^B$ is the set of all B-names $\dot{x}$ of the form $\dot{x} = \{(\dot{y}_i, b_i) \mid i \in I\}$ where I is any set, and each $\dot{y}_i$ is in V_α^B.
3. If α is a limit ordinal, or $\alpha = \mathrm{Ord}$, then $V_\alpha^B = \bigcup_{\beta\in\alpha} V_\beta^B$.
4. $V^B = V_{\mathrm{Ord}}^B$.

We define the auxiliary notion of Boolean implication by setting

$$(a \to b) = (\neg a \vee b).$$

For $\dot{x}, \dot{y} \in V^B$, we define the following *Boolean values* inductively:

- $||\dot{x} \in \dot{y}|| = \sup\{||\dot{x} = \dot{t}|| \wedge \dot{y}(\dot{t}) \mid \dot{t} \in \mathrm{dom}\, \dot{y}\}$,
- $||\dot{x} \subseteq \dot{y}|| = \inf\{\dot{x}(\dot{t}) \to ||\dot{t} \in \dot{y}|| \mid \dot{t} \in \mathrm{dom}\, \dot{x}\}$, and
- $||\dot{x} = \dot{y}|| = ||\dot{x} \subseteq \dot{y}|| \wedge ||\dot{y} \subseteq \dot{x}||$.

Verify the following properties of V^B, for all $\dot{x}$, $\dot{y}$, $\dot{z}$ in V^B.

- $||\dot{x} = \dot{x}|| = 1$.
- $||\dot{x} = \dot{y}|| \wedge ||\dot{y} = \dot{z}|| \leq ||\dot{x} = \dot{z}||$.
- $||\dot{x} \in \dot{y}|| \wedge ||\dot{x} = \dot{z}|| \leq ||\dot{z} \in \dot{y}||$,
- $||\dot{y} \in \dot{x}|| \wedge ||\dot{x} = \dot{z}|| \leq ||\dot{y} \in \dot{z}||$.

For any first order formula φ in the language of set theory, and $\dot{x}_1, \ldots, \dot{x}_n \in V^B$, we define the Boolean value of $\varphi(\dot{x}_1, \ldots, \dot{x}_n)$ inductively as follows.

- $||\neg\varphi(\dot{x}_1, \ldots, \dot{x}_n)|| = \neg||\varphi(\dot{x}_1, \ldots, \dot{x}_n)||$,
- $||(\varphi \wedge \psi)(\dot{x}_1, \ldots, \dot{x}_n)|| = ||\varphi(\dot{x}_1, \ldots, \dot{x}_n|| \wedge ||\psi(\dot{x}_1, \ldots, \dot{x}_n)||$,
- $||\exists x\, \varphi(x, \dot{x}_1, \ldots, \dot{x}_n)|| = \sup\{||\varphi(\dot{x}, \dot{x}_1, \ldots, \dot{x}_n)|| \mid \dot{x} \in V^B\}$.

For any first order formula φ in the language of set theory, and $\dot{x}_1, \ldots, \dot{x}_n \in V^B$, we say that $\varphi(\dot{x}_1, \ldots, \dot{x}_n)$ is *valid* in V^B in case $||\varphi(\dot{x}_1, \ldots, \dot{x}_n)|| = 1$.

- Every axiom of ZFC is valid in V^B. Verify this only for the axioms of Extensionality, Separation and Powerset.